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Asymptotic Bounds for Solutions to a System of Damped Integrodifferential Equations of Electromagnetic Theory*

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1. INTRODUCTION

As has been recently demonstrated [1] a system of integrodifferential equations governs the evolution of the components of the electric displacement field in a simple class of rigid holohedral isotropic dielectrics of the type introduced by Toupin and Rivlin in [2]. More specifically, we consider the following situation: Let $\Omega \subseteq \mathscr{R}^3$ be a bounded region filled with a nonconducting material dielectric substance and assume that $\partial\Omega$, the boundary of Ω, is smooth enough to admit of applications of the divergence theorem. Denote by $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{D}$, respectively, the electric field vector, the magnetic flux density, the polarization vector, and the electric displacement vector in Ω; the fields $\mathbf{E}$ and $\mathbf{D}$ are related by $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, $\epsilon_0 > 0$ a physical constant. By defining, in the usual manner, the magnetic intensity $\mathbf{H} = \mu_0^{-1}\mathbf{B}$, where $\mu_0 > 0$ satisfies $\epsilon_0\mu_0 = c^{-2}$ ($c \equiv$ speed of light in a vacuum) the differential forms of Maxwell's equations in a Lorentz reference frame (x^i, t) become

$$\begin{aligned} &\frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E} = \mathbf{0}, \qquad \operatorname{div} \mathbf{B} = 0 \\ &\operatorname{curl} \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}, \qquad \operatorname{div} \mathbf{D} = 0 \end{aligned} \tag{1.1}$$

provided that the densities of free current and free charge vanish in Ω, the magnetization is zero in Ω, and the medium is nondeformable (rigid dielectric). To obtain a determinate set of equations for the fields which appear in Maxwell's equations a set of constitutive relations among these fields must be specified and in the theory of rigid nonconducting matrial dielectrics there exists a hierachy of such constitutive assumptions of increasing complexity. The simplest con-

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stitutive assumption possible corresponds to the situation where the dielectric is a vacuum so that $\mathbf{P} = 0$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$, $\mathbf{H} = \mu_0^{-1}\mathbf{B}$. In [3, 4] this author has treated the evolution equations associated with the Maxwell–Hopkinson Dielectric in which the constitutive relation between $\mathbf{D}$ and $\mathbf{E}$ assumes the form

$$D(\mathbf{x}, t) = \epsilon \mathbf{E}(\mathbf{x}, t) + \int_{-\infty}^{t} \phi(t-\tau)\, \mathbf{E}(\mathbf{x}, \tau)\, d\tau, \qquad \epsilon > 0,$$

$$(\mathbf{x}, t) \in \Omega \times (-\infty, T), \qquad T > 0,$$

with $|\phi|$ a monotonically decreasing function. The Maxwell–Hopkinson theory retains the simple relation $\mathbf{H} = \mu_0^{-1}\mathbf{B}$ between the magnetic intensity and magnetic flux density and thus does not take into account the possible influences of magnetic memory effects. Constitutive relations generalizing those of Maxwell-Hopkinson in several directions, and allowing for an understanding of phenomena such as the Faraday effect in dielectrics, were put forth in 1960 by Toupin and Rivlin [2]. One such set of constritutive equations, for a dielectric with holohedral symmetry (i.e., a dielectric which admits the full orthogonal group as its group of material symmetry transformations) has the form

$$\begin{aligned} \mathbf{D}(\mathbf{x}, t) &= \sum_{j=0}^{n} a_j \mathbf{E}^{(j)}(\mathbf{x}, t) + \int_{-\infty}^{t} \phi(t-\tau)\, \mathbf{E}(\mathbf{x}, \tau)\, d\tau \\ \mathbf{H}(\mathbf{x}, t) &= \sum_{j=0}^{n} b_j \mathbf{B}^{(j)}(\mathbf{x}, t) + \int_{-\infty}^{t} \psi(t-\tau)\, \mathbf{B}(\mathbf{x}, \tau)\, d\tau \end{aligned} \tag{1.2}$$

where the superscripts denote differentiation with respect to the time parameter and the coefficients a_j, b_j are constants; whereas equations (3.2) still effect an a priori separation of electric and magnetic effects they now allow for consideration of dielectric materials exhibiting magnetic memory and may be viewed as a linearized version of a more general theory introduced by Volterra in 1912 [5] to treat the case where the dielectric substance is anisotropic, nonlinear, and magnetized, viz:

$$\begin{aligned} \mathscr{D}(\mathbf{x}, t) &= \epsilon \cdot \mathbf{E}(\mathbf{x}, t) + \underset{-\infty}{\overset{t}{\mathscr{D}}}(\mathbf{E}(\mathbf{x}, \tau)) \\ \mathbf{B}(\mathbf{x}, t) &= \boldsymbol{\mu} \cdot \mathbf{H}(\mathbf{x}, t) + \underset{-\infty}{\overset{t}{\mathscr{B}}}(\mathbf{H}(\mathbf{x}, \tau)) \end{aligned} \tag{1.3}$$

where ϵ, μ are constant second-order tensors; the constitutive relations (1.2) follow from the set delineated in (1.3) when, among other assumptions, it is assumed that the functionals $\mathscr{D}$, $\mathscr{B}$ are linear and isotropic.

In [1] we have studied various consequences of the constitutive hypothesis

(1.2) under the simplifying assumptions that $a_j = b_j = 0$, $j \geqslant 1$ and that the past histories of the electric and magnetic fields are of the form

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \mathbf{0}, & -\infty < t \leqslant -t_h\,, \\ &= \mathbf{E}_h(\mathbf{x}, t), & -t_h < t < 0; \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{0}, & -\infty < t \leqslant -t_h\,, \\ &= \mathbf{B}_h(\mathbf{x}, t), & -t_h < t < 0, \end{aligned} \tag{1.4}$$

for some $t_h > 0$. In particular for memory functions ϕ, ψ which are sufficiently smooth on $(-t_h\,, \infty)$ we have the following

LEMMA [1]. *The evolution of the electric displacement field* $\mathbf{D}(\mathbf{x}, t)$ *in any holohedral isotropic dielectric* (*which conforms to the constitutive hypothesis* (1.2) *with* $a_j = b_j = 0$, $j \geqslant 1$ *and past histories of the form* (1.4)) *for some* $t_h > 0$, *is governed by a system of damped integrodifferential equations of the form*

$$\begin{aligned} &\frac{\partial^2 D_i}{\partial t^2} + \Psi(0)\frac{\partial D_i}{\partial t} + \Psi(0)\,[D_i - c_0\nabla^2 D_i] \\ &\quad + \int_{-t_h}^{t} \left[\Psi(t-\tau)\, D_i(\tau) - \left(\frac{b_0}{a_0}\right)\Phi(t-\tau)\,\nabla^2 D_i(\tau)\right] d\tau \\ &= 0, \quad in \quad \Omega, \qquad i = 1, 2, 3, \quad c_0 = b_0/a_0\dot{\psi}(0) \end{aligned} \tag{1.5}$$

provided $\mathbf{D}_h{}^+(\mathbf{x}, -t_h) = \mathbf{0}$ *in* Ω *and* $\dot{\Psi}(0) \neq 0$.

In (1.5) $\Phi(t)$ is given in terms of the memory function $\phi(t)$ via the recursion relations

$$\begin{aligned} &\Phi(t) = \sum_{n=1}^{\infty} (-1)^n \phi^n(t), \qquad t \geqslant 0, \\ &\phi^1(t) = a_0^{-1}\phi(t), \qquad \phi^n(t) = \int_{-t_h}^{t} \phi^1(t-\tau)\,\phi^{n-1}(\tau)\,d\tau, \end{aligned} \tag{1.6}$$

for $n \geqslant 2$, with a similar definition for $\Psi(t)$ in terms of $\psi(t)$. We assume that $a_0 > 0$, $b_0 > 0$; it can be shown that $\Psi(0) = -b_0^{-1}\psi(0)$ and thus we assume $\psi(0) < 0$ so that the coefficient of $\partial D_i/\partial t$ in (1.5), i.e., $\Psi(0) > 0$.

Remark. The system of integrodifferential equations (1.5), for the components of the electric displacement field, is obtained by combining the constitutive relations (1.2) (with $a_j = b_j = 0$, $j \geqslant 1$ and past histories of the form (1.4)) with the inverted constitutive equations, giving $\mathbf{E}$ and $\mathbf{B}$ in terms of $\mathbf{D}$ and $\mathbf{H}$, respectively, Maxwell's equations (1.1), and the vector identity

$$\Delta\mathbf{V}(\mathbf{x}) = \operatorname{grad}(\operatorname{div}\mathbf{V}(\mathbf{x})) - \operatorname{curl}\operatorname{curl}\mathbf{V}(\mathbf{x})$$

which is valid $\forall \mathbf{x} \in \Omega$ for any vector field $\mathbf{V}(\cdot)$ which is sufficiently smooth on Ω; the constitutive relations (1.2) are inverted by the usual technique of successive approximations. For the details of the computation we refer to [1, Sect. 3].

We now formulate, in a bounded domain $\tilde{\Omega} \supset \Omega$, an initial-history boundary value problem for the components of the electric displacement field: Let $\hat{\Omega} \subseteq \mathscr{R}^3$ be a bounded domain such that $\Omega \subset \hat{\Omega}$; we assume that the region $\hat{\Omega}/\Omega$ is occupied by a perfect conductor so that $\mathbf{D} = \mathbf{0}$ in $\hat{\Omega}/\Omega$ [6, Sect. 10.5]). On $\partial\Omega$, $\mathbf{D}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \sigma(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$, where $\mathbf{n}(\mathbf{x})$ is the unit outward normal to $\partial\Omega$ at $\mathbf{x}$ and $\sigma(\mathbf{x})$ is the free charge density at $\mathbf{x} \in \partial\Omega$. Now let $\tilde{\Omega}$ be any bounded domain in $\mathscr{R}^3$ satisfying $\Omega \subset \tilde{\Omega} \subseteq \hat{\Omega}$; then for $(\mathbf{x}, t) \in \partial\tilde{\Omega} \times (-t_h, \infty)$, $\mathbf{D}(\mathbf{x}, t) = \mathbf{0}$. We have, or course, Eqs. (1.5) in $\Omega \times (0, \infty)$ and $\mathbf{D} = \mathbf{0}$ in $\tilde{\Omega}/\Omega \times (-t_h, \infty)$. In conjunction with these equations and the prescription of the past history for $(\mathbf{x} \in \Omega)$ given by

$$\begin{aligned} \mathbf{D}(\mathbf{x}, t) &= \mathbf{0}, && -\infty < t < -t_h, \\ &= \mathbf{D}_h(\mathbf{x}, t), && -t_h \leqslant t < 0, \end{aligned} \tag{1.7a}$$

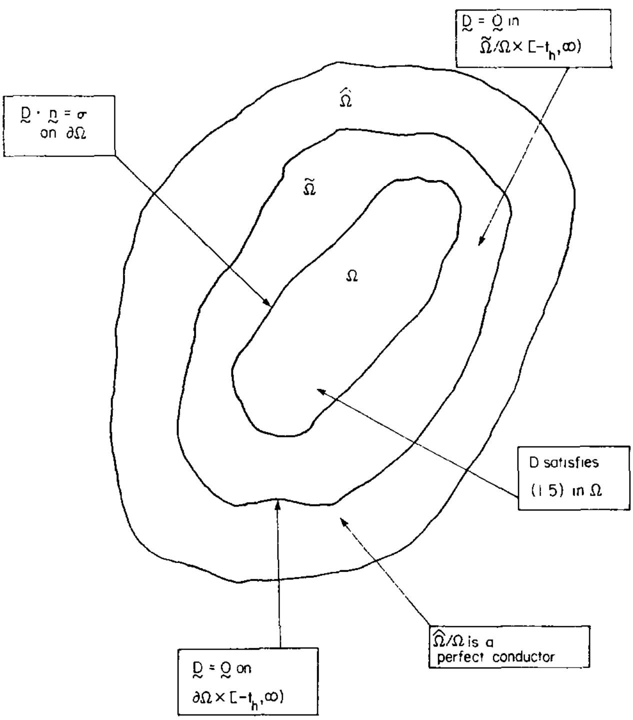

FIG. 1. Holohedral isotropic dielectric embedded in a perfect conductor.

we consider initial data of the form

$$\begin{aligned} \mathbf{D}(\mathbf{x}, 0) &= D_0(\mathbf{x}), \qquad \mathbf{x} \in \tilde{\Omega}, \\ \mathbf{D}_t(\mathbf{x}, 0) &= \mathbf{D}_1(\mathbf{x}), \qquad \mathbf{x} \in \tilde{\Omega}, \end{aligned} \tag{1.7b}$$

where $\mathbf{D}_0(\mathbf{x}) = 0$ in $\tilde{\Omega}/\Omega$, $\mathbf{D}_1(\mathbf{x}) = \mathbf{0}$ in $\tilde{\Omega}/\Omega$ and we assume that $\int_\Omega (\mathbf{D}_0)_i (\mathbf{D}_0)_i \, d\mathbf{x} \neq 0$. The situation is depected in Fig. 1.

2. The Initial-History Value Problem in Hilbert Spacf

We introduce three spaces: $H = \mathbf{L}_2(\tilde{\Omega})$[1] with the standard inner-product

$$\langle \mathbf{v}, \mathbf{w} \rangle_{L_2} = \int_{\tilde{\Omega}} v_i w_i \, d\mathbf{x}$$

the Sobolev space $H_+ = \mathbf{H}_0^1(\tilde{\Omega})$ with inner-product

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{H}_0^1} = \int_{\tilde{\Omega}} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\mathbf{x}$$

and $H_- = \mathbf{H}^{-1}(\tilde{\Omega})$, the completion of $C_0^\infty(\tilde{\Omega})$ under the norm

$$\| \mathbf{v} \|_{\mathbf{H}^{-1}} = \sup_{w \in \mathbf{H}_0^1} \left[\left| \int_{\tilde{\Omega}} v_i w_i \, d\mathbf{x} \right| \Big/ \left(\int_{\tilde{\Omega}} \frac{\partial w_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\mathbf{x} \right)^{1/2} \right].$$

It is well-known that $\mathbf{H}^{-1} = (\mathbf{H}_0^1)^{-1}$ (dual space) that $\mathbf{H}_0^1(\tilde{\Omega}) \subseteq \mathbf{L}_2(\tilde{\Omega})$, both topologically and algebraically, and that $\mathbf{H}_0^1(\tilde{\Omega})$ is dense in $\mathbf{L}_2(\tilde{\Omega})$; we denote the embedding constant for the inclusion map $i: \mathbf{H}_0^1(\tilde{\Omega}) \to \mathbf{L}_2(\tilde{\Omega})$ by γ, so that $\| \mathbf{v} \|_{\mathbf{L}_2}(\tilde{\Omega}) \leqslant \gamma \| \mathbf{v} \|_{\mathbf{H}_0^1}(\tilde{\Omega})$, $\forall \mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$. Operators $\mathbf{N} \in \mathscr{L}_s(\mathbf{H}_0^1(\tilde{\Omega}); \mathbf{H}^{-1}(\tilde{\Omega}))$ and $\mathbf{K} \in L^2((-\infty, \infty); \mathscr{L}_s(\mathbf{H}_0^1(\tilde{\Omega}), \mathbf{H}^{-1}(\tilde{\Omega}))$, where $\mathscr{L}_s(\mathbf{H}_0^1(\tilde{\Omega}); \mathbf{H}^{-1}(\tilde{\Omega}))$ denotes the space of all bounded symmetric linear operators from $\mathbf{H}_0^1(\tilde{\Omega})$ into $\mathbf{H}^{-1}(\tilde{\Omega})$, may now be defined as follows: for any $\mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$, $t \in (-\infty, \infty)$

$$(\mathbf{N}\mathbf{v})_i = \Psi(0) [c_0 \nabla^2 v_i - v_i], \qquad c_0 \equiv b_0/a_0\Psi(0),$$

$$(\mathbf{K}(t)\, \mathbf{v})_i = \Psi(t)\, v_i - \left(\frac{b_0}{a_0}\right) \Phi(t)\, \nabla^2 v_i$$

wherewhere the derivatives are understood in the distribution sense, i.e., $\nabla^2 v_i = \nu_i \in \mathscr{L}_2(\tilde{\Omega})$ is such that for any $\phi \in C_0^\infty(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} \phi \nu_i \, d\mathbf{x} = - \int_{\tilde{\Omega}} \frac{\partial \phi}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x}.$$

[1] $\mathbf{L}_2(\Omega) = (L_2(\tilde{\Omega}))^3$, i.e., $\mathbf{v} \in \mathbf{L}_2(\tilde{\Omega})$ iff $v_i \in L_2(\tilde{\Omega})$, $i = 1, 2, 3$, with similar interpretations for $\mathbf{H}_0^1(\tilde{\Omega})$, $\mathbf{H}^{-1}(\tilde{\Omega})$ introduced below.

The symmetry and boundedness of $\mathbf{N}$ and $\mathbf{K}(t)$, $t \in (-\infty, \infty)$, as maps of $\mathbf{H}_0^1(\tilde{\Omega})$ into $\mathbf{H}^{-1}(\tilde{\Omega})$ will be verified in Section 3. If we now set $\Gamma = \Psi(0) > 0$ then with the definitions of $\mathbf{N}$, $\mathbf{K}(t)$ as given above the initial-boundary value problem (1.5)–(1.7) is equivalent to the following initial-history value problem in Hilbert space: find $\mathbf{u} \in C^2([0, \infty); \mathbf{H}_0^1(\tilde{\Omega}))$ such that $\mathbf{u}_t \in C^1([0, \infty); \mathbf{H}_0^1(\tilde{\Omega}))$, $\mathbf{u}_{tt} \in C([0, \infty); \mathbf{H}^{-1}(\tilde{\Omega}))$[2] and

$$
\begin{aligned}
\mathbf{u}_{tt} + \Gamma \mathbf{u}_t - \mathbf{N}\mathbf{u} + \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau = \mathbf{0}, \qquad t > 0,& \\
\mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{u}_t(0) = \mathbf{v}_0(\mathbf{u}_0, \mathbf{v}_0 \in \mathbf{H}_0^1(\tilde{\Omega})),& \\
\mathbf{u}(\tau) = \mathbf{0}, \qquad -\infty < \tau < -t_h,& \\
= \mathbf{U}_h(\tau), \qquad -t_h \leqslant \tau < 0,&
\end{aligned}
\tag{2.1}
$$

In general, without definiteness assumptions on the operators $\mathbf{N}$ and $\mathbf{K}(t)$, $t \in (-\infty, \infty)$, this abstract initial-history value problem for $\mathbf{u}(t)$ is ill-posed. However, we will show that with no definiteness assumptions on $\mathbf{N}$ and only mild assumptions on $\mathbf{K}(t)$, i.e.,

(A1) $-\langle \mathbf{v}, \mathbf{K}(0)\,\mathbf{v} \rangle \geqslant 0$, $\qquad \mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$

(A2) $\mathscr{K}(t) = \| \mathbf{K}(t) \|_{\mathscr{L}_s(\mathbf{H}_0^1; \mathbf{H}^{-1})}$ $\qquad$ satisfies $\mathscr{K}(\cdot) \in L_1[0, \infty)$

(A3) $\hat{\mathscr{K}}(t) = \int \| \mathbf{K}_t \|_{\mathscr{L}_s(\mathbf{H}_0^1(\Omega); \mathbf{H}^{-1}(\Omega))}\, dt$ $\qquad$ satisfies $\hat{\mathscr{K}}(\cdot) \in L_1[0, \infty)$ with $\hat{\mathscr{K}}(0) = 0$.

where $\mathbf{K}_t$ denotes the strong operator derivative of $\mathbf{K}$, it is possible to derive asymptotic lower bounds for the $\mathbf{L}_2$ norms of solutions $\mathbf{u}$ to the system (2.1) which lie in classes of bounded perturbations $\mathscr{N}$ of the form

$$
\mathscr{N} = \{\mathbf{v} \in C([-t_h, \infty); \mathbf{H}_0^1), \sup_{[-t_h, \infty)} \| \mathbf{v} \|_{\mathbf{H}_0^1} \leqslant N\}
\tag{2.2}
$$

for some $N > 0$. Our results are obtained by using a mixture of logarithmic convexity and concavity arguments which have been used successfully now for over a period of more than a decade in order to treat problems of uniqueness, stability, and continuous dependence for solutions to ill-posed initial-boundary value problems and initial-history boundary value problems associated with various linear and non-linear partial differential equations and integrodifferential equations [see [7–12], and the references cited therein].

[2] $\mathbf{u}: [0, \infty) \to \mathbf{H}_0^1$ satisfying these smoothness assumptions will be called a strong solution of (2.1).

Remarks. We offer below some comments regarding previous work related to one or more aspects of the current investigation:

(i) Growth estimates for a class of damped linear integradifferential equations associated with holohedral isotropic dielectric response have been obtained in [1] via a concavity argument; the nature of the estimates precludes our obtaining from them any information concerning the behavior of solutions as $t \to +\infty$. More specifically, we have shown the following: For any $\alpha > 0$, let $\mathbf{u}^\alpha$ be a strong solution of (2.1) with $\mathbf{u}^\alpha(0) = \alpha \mathbf{u}_0$, where it is assumed that $\langle \mathbf{u}_0, \mathbf{v}_0 \rangle_{\mathbf{L}_2} > 0$, $\langle \mathbf{u}_0, \mathbf{N}\mathbf{u}_0 \rangle_{\mathbf{L}_2} > 0$, $\langle \mathbf{u}_0, \int_{t_n}^0 \mathbf{K}(-\tau)\, \mathbf{U}_h(\tau)\, d\tau \rangle_{\mathbf{L}_2} < 0$, and that $\mathbf{K}$ satisfies hypotheses A1, A2, and $\int_0^\infty \| \mathbf{K}_t \|_{\mathscr{L}_s(\mathbf{H}_0^1(\Omega)\mathbf{H}^{-1}(\Omega))}\, dt < \infty$. Then, provided $\| \mathbf{u}_0 \|_{\mathbf{L}_2}^2 \leqslant (2/\Gamma) \langle \mathbf{u}_0, \mathbf{v}_0 \rangle_{\mathbf{L}_2}$ and

$$T > \frac{1}{\Gamma} \ln \left(\frac{2\langle \mathbf{u}_0, \mathbf{v}_0 \rangle_{\mathbf{L}_2}}{2\langle \mathbf{u}_0, \mathbf{v}_0 \rangle_{\mathbf{L}_2} - \Gamma \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2} \right),$$

it follows that

$$\sup_{-\infty < t < T} \| \mathbf{u}^\alpha(t) \|_{\mathbf{H}_0^1(\Omega)} \geqslant \frac{|\langle \mathbf{u}_0, \int_{t_h}^0 \mathbf{K}(-\tau)\, \mathbf{U}_h(\tau)\, d\tau \rangle_{L_2}|^{1/2}}{\gamma \Sigma_T}\, \alpha^{1/2} \tag{2.3}$$

for each $\alpha \geqslant \| \mathbf{v}_0 \|_{\mathbf{L}_2} / \langle \mathbf{u}_0, \mathbf{N}\mathbf{u}_0 \rangle_{\mathbf{L}_2}^{1/2}$, where

$$\Sigma_T = \tfrac{1}{2} \| \mathbf{N} \|_{\mathscr{L}_s(\mathbf{H}_0^1(\Omega);\mathbf{H}^{-1}(\Omega))} + \int_0^\infty \| \mathbf{K}(\tau) \|_{\mathscr{L}_s(\mathbf{H}_0^1(\Omega);\mathbf{H}^{-1}(\Omega))}\, d\tau$$
$$+ T \int_0^\infty \| \mathbf{K}_\tau(\tau) \|_{\mathscr{L}_s(\mathbf{H}_0^1(\Omega);\mathbf{H}^{-1}(\Omega))}\, d\tau\,.$$

Estimate (2.3) does not require that $\mathbf{u}$ belong to a class of bounded pertubations of the type specified by the set $\mathscr{N}$ defined in (2.2) but it is limited to $T < \infty$.[3] An estimate completely analogous to (2.3) is available for the undamped situation, i.e., (2.1) with $\Gamma = 0$, but can not, in view of the hypotheses which led to (2.3) for the undamped situation, be obtained by simply setting $\Gamma = 0$ in those hypotheses. The initial-history value problem (2.1), with $\Gamma = 0$, is shown in [3] to model the evolution of the electric displacement field $\mathbf{D}$ in a nonconducting dielectric of Maxwell–Hopkinson type and an estimate of the type (2.3) is obtained there under the assumption that $T > \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2 / (2\langle \mathbf{u}_0, \mathbf{v}_0 \rangle_{\mathbf{L}_2})$.

Finally, we indicate that in contrast to the various concavity arguments employed in [1, 3], for the damped and undamped integrodifferential initial-history value problems associated with (2.1), growth estimates for solutions to these respective problems which lie in bounded classes of perturbations, of the type $\mathscr{N}$, can also be obtained by using logarithmic convexity arguments, i.e., [4]; the nature of the logarithmic convexity argument, however, involves not

[3] The estimates in [1, 3] are obtained by using a modified concavity argument.

only a restriction to classes of bounded perturbations but also a restriction to finite time intervals of the form $[0, T)$, $T < \infty$, and requires, in addition, the stronger hypothesis that

$$-\langle \mathbf{v}, \mathbf{K}(0)\, \mathbf{v}\rangle_{\mathbf{L}_2} \geqslant \kappa \| \mathbf{v} \|^2_{\mathbf{H}_0^1}, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$$

with

$$\kappa \geqslant \gamma T \sup_{[0,T)} \| \mathbf{K}_t \|_{\mathscr{L}_s(\mathbf{H}_0^1(\tilde{\Omega});\mathbf{H}^{-1}(\tilde{\Omega}))}.$$

Logarithmic convexity arguments have also been employed in [8, 9], to obtain uniqueness and continuous dependence theorems, as well as growth estimates, for solutions to ill-posed initial-history boundary value problems in isothermal viscoelasticity, and in [16] to obtain growth estimates for solutions to a class of nonlinear integrodifferential equations in Hilbert space.

(ii) Several authors ([13, 14], and the references cited theorem) have studied the asymptotic behavior of solutions to nitial-value problems associated with damped evolution equations of the form

$$\mathbf{u}_{tt} + \mathbf{A}\mathbf{u}_t + \mathbf{B}\mathbf{u} = 0 \tag{2.4}$$

where $\mathbf{u}\colon [0, \infty) \to H$, a real Hilbert space with inner-product $\langle\ ,\ \rangle$ and natural norm $\|(\cdot)\|$; the usual assumptions which are made are that $\mathbf{B}$ is in $\mathscr{L}(H; H)$ and satisfies a coerciveness condition of the form

$$\langle \mathbf{v}, \mathbf{B}\mathbf{v}\rangle \geqslant \lambda \| \mathbf{v} \|^2, \qquad \lambda > 0, \quad \mathbf{v} \in \mathscr{D}(\mathbf{B}) \tag{2.5}$$

with $\overline{\mathscr{D}(\mathbf{B})} = H$. When the linear operator $\mathbf{A}$ satisfies $\langle \mathbf{A}\mathbf{v}, \mathbf{v}\rangle \geqslant 0$, and $\mathbf{A}^{-1}$ exists (the strongly damped case) it is well-known that the energy

$$\mathscr{E}(t) = \tfrac{1}{2}(\| \mathbf{u}_t \|^2 + \langle \mathbf{u}(t), \mathbf{B}\mathbf{u}(t)\rangle)$$

decays at a uniform exponential rate; even if $\mathbf{A}^{-1}$ does not exist (the weakly damped case) it can be shown that in certain circumstances $\lim_{t\to\infty} \mathscr{E}(t) = 0$. In [15] we considered the system

$$\begin{aligned} &\mathbf{u}^\alpha_{tt} + \Gamma \mathbf{u}^\alpha_t - N\mathbf{u}^\alpha = 0, \qquad \Gamma > 0, \quad 0 \leqslant t < \infty, \\ &\mathbf{u}^\alpha(0) = \alpha \mathbf{u}_0, \qquad \mathbf{u}_t^\alpha(0) = \mathbf{v}_0 \qquad (\mathbf{u}_0, \mathbf{v}_0 \in \mathscr{D}(\mathbf{N})) \end{aligned} \tag{2.6}$$

with $\alpha > 0$ and $\mathbf{u}^\alpha \in C^2([0, \infty); \mathscr{D}(N))$. If $\langle \mathbf{v}, \mathbf{N}\mathbf{v}\rangle \leqslant -\lambda \| \mathbf{v} \|^2$, $\lambda > 0$, $\forall \mathbf{v} \in \mathscr{D}(\mathbf{N})$ (the hypothesis corresponding to (2.5)) asymptotic stability in the energy norm follows immediately; however, it is shown [15] that if $\mathbf{N}$ is symmetric, $\langle \mathbf{v}, \mathbf{N}\mathbf{v}\rangle \geqslant 0$, $\forall \mathbf{v} \in D(\mathbf{N})$, and there exists an element $\hat{\mathbf{u}}_0 \in \mathscr{D}(\mathbf{N})$ such that $\langle \hat{\mathbf{u}}_0, \mathbf{N}\hat{\mathbf{u}}_0\rangle > 0$, any solution of (2.6) having the requisite smoothness must satisfy, for $\mathbf{u}_0 = \hat{\mathbf{u}}_0$, and α sufficiently large

$$\lim_{t\to\infty} \| \mathbf{u}^\alpha(t)\|^2 \geqslant \alpha^2 \| \hat{\mathbf{u}}_0 \|^2 e^{-\Sigma_0(\alpha,\Gamma)} \tag{2.7}$$

where $\Sigma_0(\alpha, \Gamma)$ depends on $\hat{\mathbf{u}}_0$, v_0 and satisfies $\lim_{\Gamma\to\infty} \Sigma_0(\alpha, \Gamma) = 0$ (i.e., solutions are asymptotically bounded away from zero, for α sufficiently large, no matter how strong the damping is). The asymptotic lower bound (2.7) is obtained in [15] by employing a mixture of logarithmic concavity and convexity arguments to establish the estimate

$$\| \mathbf{u}^\alpha(t)\|^2 \geqslant \alpha^2 \| \hat{\mathbf{u}}_0 \|^2 \exp\left[\left\{\frac{\langle \hat{\mathbf{u}}_0 , \mathbf{v}_0\rangle}{\alpha\Gamma \| \hat{\mathbf{u}}_0 \|^2}\right\} (1 - e^{-\Gamma t})\right] \tag{2.8}$$

for all $t \geqslant 0$, $\alpha \geqslant \| \mathbf{v}_0 \|/(\langle \hat{\mathbf{u}}_0, \mathbf{N}\hat{\mathbf{u}}_0\rangle)^{1/2}$, and does not require that $\mathbf{u}^\alpha$ be a priori restricted to lie in a class of bounded perturbations; estimate (2.8) may be easily extended to the case where $\mathbf{N} \in \mathscr{L}_s(H_+ , H_-)$, $\mathbf{u}^\alpha$: $[0, \infty) \to H_+$, where H_+, is a second Hilbert space with inner product $\langle\ ,\ \rangle_+$ and natural norm $\|(\cdot)\|_+$ such that $H_+ \subseteq H$, both algebraically and topologically, and H_- is the completion of H under the norm $\|(\cdot)\|_-$ defined via

$$\| \mathbf{w} \|_- = \sup_{\mathbf{v}\in \mathbf{H}_+} \frac{|\langle \mathbf{v}, \mathbf{w}\rangle|}{\| \mathbf{v} \|_+}.$$

In particular, system (2.1) reduces to (2.6) if $\mathbf{K} = \mathbf{0}$, $\mathbf{u}_0 \to \alpha\mathbf{u}_0$, and we identify $H = \mathbf{L}_2(\tilde{\Omega})\ H_+ = \mathbf{H}_0^1(\tilde{\Omega})$, $H_- = \mathbf{H}^{-1}(\tilde{\Omega})$. For system (2.1) we shall derive asymptotic lower bounds of the form (2.7) without introducing a one-parameter family of initial-data functions of the form $\alpha\mathbf{u}_0$, and without making any definiteness assumptions on $\mathbf{N}$. For definiteness hypothesis on $\mathbf{N}$, $\mathbf{K}(t)$, which imply the existence, uniqueness, and asymptotic stability of solutions to initial-history value problems of the type (2.1) we refer the reader to [16, 17] and the references cited therein.

3. Asymptotic Lower Bounds for Solutions

We want to show that, under an appropriate set of circumstances, solutions $\mathbf{u} \in \mathscr{N}$ of system (2.1) are asymptotically bounded away from zero, in the $\mathbf{L}_2$ norm, even as the damping term $\Gamma \to +\infty$. To this end we will establish the following:

Theorem. *Let* $\mathbf{u} \in \mathscr{N}$ *be a strong solution of* (2.1) *where* $\mathbf{N} \in \mathscr{L}_s(\mathbf{H}_0^1; \mathbf{H}^{-1})$ *and* $\mathbf{K} \in L^2((-\infty, \infty); \mathscr{L}_s(\mathbf{H}_0^1, \mathbf{H}^{-1}))$ *such that hypotheses* A1, A2, *and* A3 (*of Section* 1) *are satisfied. If* $\mathscr{E}(0) = \frac{1}{2} \| \dot{\mathbf{v}}_0 \|_{\mathbf{L}_2}^2 - \langle \mathbf{u}_0 , \mathbf{N}\mathbf{u}_0\rangle_{\mathbf{L}_2} < 0$ *with*

$$| \mathscr{E}(0)| > \tfrac{3}{2}\gamma N^2[\| \mathscr{K} \|_{L_1[0,\infty)} + \| \mathscr{K} \|_{L_1[0,\infty)}] \tag{3.1}$$

then for all t, $0 \leqslant t < \infty$, *and any* $\beta > 0$, $F(t) = \| \mathbf{u} \|_{\mathbf{L}_2}^2$ *satisfies the differential inequality*

$$FF'' - \left(\frac{\beta + 1}{2\beta + 1}\right) F'^2 \geqslant -\Gamma F F'. \tag{3.2}$$

Proof. From the definition of $F(t)$ we have $F' = 2\langle \mathbf{u}, \mathbf{u}_t \rangle_{\mathbf{L}_2}$ and $F'' = 2 \| \mathbf{u}_t \|_{\mathbf{L}_2}^2 + 2\langle \mathbf{u}, \mathbf{u}_{tt} \rangle_{\mathbf{L}_2}$. Direct computation then yields

$$FF'' - (\beta + 1) F'^2 = 4(\beta + 1)\, S_\beta{}^2 + 2F\{\langle \mathbf{u}, \mathbf{u}_{tt} \rangle_{\mathbf{L}_2} - (2\beta + 1) \| \mathbf{u}_t \|_{\mathbf{L}_2}^2\} \tag{3.3}$$

where

$$S_\beta{}^2(t) = \| \mathbf{u} \|_{\mathbf{L}_2}^2 \| \mathbf{u}_t \|_{\mathbf{L}_2}^2 - \langle \mathbf{u}, \mathbf{u}_t \rangle_{\mathbf{L}_2} \geqslant 0 \tag{3.4}$$

by the Schwarz inequality. Therefore, for $0 \leqslant t < \infty$, and any $\beta > 0$

$$FF'' - (\beta + 1) F'^2 \geqslant 2FG_\beta \tag{3.5}$$

where, in view of the integrodifferential equation (2.1_1) for $\mathbf{u}(t)$

$$\begin{aligned} G_\beta(t) = \langle \mathbf{u}, \mathbf{N}\mathbf{u} \rangle_{\mathbf{L}_2} - \Gamma \langle \mathbf{u}, \mathbf{u}_t \rangle_{\mathbf{L}_2} - (2\beta + 1) \| \mathbf{u}_t \|_{\mathbf{L}_2}^2 \\ - \left\langle \mathbf{u}, \int_{-\infty}^t \mathbf{K}(t - \tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}_2} \end{aligned} \tag{3.6}$$

As $F'(t) = 2\langle \mathbf{u}, \mathbf{u}_t \rangle_{\mathbf{L}_2}$ we may rewrite (3.6) as

$$\begin{aligned} G_\beta(t) &= -\frac{\Gamma}{2} F' - (2\beta + 1)\, [\| \mathbf{u}_t \|_{\mathbf{L}_2}^2 - \langle \mathbf{u}, \mathbf{N}\mathbf{u} \rangle_{\mathbf{L}_2}] \\ &\quad - 2\beta \langle \mathbf{u}, \mathbf{N}\mathbf{u} \rangle_{\mathbf{L}_2} - \left\langle \mathbf{u}, \int_{-\infty}^t \mathbf{K}(t - \tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}_2} \\ &= -\frac{\Gamma}{2} F' - 2(2\beta + 1)\, \mathscr{E}(t) - 2\beta \langle \mathbf{u}, \mathbf{N}\mathbf{u} \rangle_{\mathbf{L}_2} \\ &\quad - \left\langle \mathbf{u}, \int_{-\infty}^t \mathbf{K}(t - \tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}_2} \end{aligned} \tag{3.7}$$

in view of the definition of $\mathscr{E}(t)$. Taking the $\mathbf{L}_2$ inner-product of (2.1_1) with $\mathbf{u}_t$ and integrating we easily obtain

$$\mathscr{E}(t) = \mathscr{E}(0) - \Gamma \int_0^t \| \mathbf{u}_\tau \|_{\mathbf{L}_2}^2\, d\tau - \int_0^t \left\langle \mathbf{u}_\tau, \int_{-\infty}^\tau \mathbf{K}(\tau - \lambda)\, \mathbf{u}(\lambda)\, d\lambda \right\rangle_{\mathbf{L}_2} d\tau \tag{3.8}$$

and substitution into (3.7_2) then yields

$$
\begin{aligned}
G_\beta(t) \geq & -\frac{\Gamma}{2}F' - 2(2\beta+1)\,\mathscr{E}(0) - 2\beta\langle \mathbf{v}, \mathbf{N}\mathbf{u}\rangle_{\mathbf{L}_2} \\
& + 2(2\beta+1)\int_0^t \left\langle \mathbf{u}_\tau, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda\right\rangle_{\mathbf{L}_2} d\tau \qquad (3.9)\\
& - \left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau\right\rangle_{\mathbf{L}_2},
\end{aligned}
$$

where we have dropped a non-negative term proportional to $\int_0^t \|\mathbf{u}_\tau\|_{\mathbf{L}_2}\,d\tau$. If we now take the $\mathbf{L}_2$ inner-product of (2.1_1) with $\mathbf{u}(t)$ and use the definition of $F(t)$ we obtain the identity

$$
\frac{1}{2}F'' + \frac{\Gamma}{2}F' = \|\mathbf{u}_t\|_{\mathbf{L}_2}^2 + \langle \mathbf{u}, \mathbf{N}\mathbf{u}\rangle_{\mathbf{L}_2} - \left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau\right\rangle_{\mathbf{L}_2}. \qquad (3.10)
$$

which implies that

$$
-2\beta\langle \mathbf{u}, \mathbf{N}\mathbf{u}\rangle_{\mathbf{L}_2} = -\beta F'' - \beta\Gamma F' + 2\beta\|\mathbf{u}_t\|_{\mathbf{L}_2}^2 - 2\beta\left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau\right\rangle_{\mathbf{L}_2}. \qquad (3.11)
$$

Substituting from (3.11) into (3.9), collecting terms, and dropping a non-negative expression proportional to $\|\mathbf{u}_t\|_{\mathbf{L}_2}^2$ now yields the following estimate for $G_\beta(t)$:

$$
\begin{aligned}
G_\beta(t) \geq & -\Gamma(\beta+\tfrac{1}{2})F' - \beta F'' - 2(2\beta+1)\,\mathscr{E}(0) \\
& - (2\beta+1)\left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau\right\rangle_{\mathbf{L}_2} \qquad (3.12)\\
& + 2(2\beta+1)\int_0^t \left\langle \mathbf{u}_\tau, \int_{-\infty}^{t} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda\right\rangle_{\mathbf{L}_2} d\tau.
\end{aligned}
$$

Substitution for $G_\beta(t)$ from (3.12) into the differential inequality (3.5) now produces

$$
\begin{aligned}
FF'' - (\beta+1)F'^2 \geq & -2\Gamma(\beta+\tfrac{1}{2})FF' - 2\beta FF'' - 4(2\beta+1)\,\mathscr{E}(0)F \\
& - 2(2\beta+1)F\left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau\right\rangle_{\mathbf{L}_2} \qquad (3.13)\\
& + 4(2\beta+1)F\int_0^t \left\langle \mathbf{u}_\tau, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda\right\rangle_{\mathbf{L}_2} d\tau
\end{aligned}
$$

which is equivalent to

$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right) F'^2 \geqslant -\Gamma FF' - 4F\mathscr{E}(0) - 2F\left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau \right\rangle_{\mathbf{L}_2}$$
$$+ 4F\int_0^t \left\langle \mathbf{u}_\tau, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda \right\rangle_{\mathbf{L}_2} d\tau \tag{3.14}$$

or, in view of our hypotheses that $\mathscr{E}(0) < 0$

$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right) F'^2 \geqslant -\Gamma FF' + 2F\Big[2\,|\,\mathscr{E}(0)| - \left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau \right\rangle_{\mathbf{L}_2}$$
$$+ 2\int_0^t \left\langle \mathbf{u}_\tau, \int_{-\infty}^{t} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\tau \right\rangle_{\mathbf{L}_2}\Big]\,d\tau. \tag{3.15}$$

We now seek to bound the two expressions involving $\mathbf{K}(t)$ on the right-hand side of (3.15). Let us first note, however, that as

$$\left\langle \mathbf{u}_\tau, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\tau \right\rangle_{\mathbf{L}_2} = \frac{d}{d\tau}\left\langle \mathbf{u}(\tau), \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda \right\rangle_{\mathbf{L}_2}$$
$$- \left\langle \mathbf{u}(\tau), \int_{-\infty}^{\tau} \mathbf{K}_\tau(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda \right\rangle_{\mathbf{L}_2} \tag{3.16}$$
$$- \langle \mathbf{u}(\tau), \mathbf{K}(0)\,\mathbf{u}(\tau)\rangle_{\mathbf{L}_2}$$

(3.15) has the equivalent form

$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right) F'^2 \geqslant -\Gamma FF' + 2F\Big[2\,|\,\mathscr{E}(0)| - 2\int_0^t \langle \mathbf{u}(\tau), \mathbf{K}(0)\,\mathbf{u}(\tau)\rangle_{\mathbf{L}_2}\,d\tau$$
$$- 2\int_0^t \left\langle \mathbf{u}(\tau), \int_{-\infty}^{\tau} \mathbf{K}_\tau(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda \right\rangle_{\mathbf{L}_2} d\tau$$
$$- 2\left\langle \mathbf{u}_0, \int_{-\infty}^{0} \mathbf{K}(-\tau)\,\mathbf{u}(\tau)\,d\tau \right\rangle_{\mathbf{L}_2}$$
$$+ \left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau \right\rangle_{\mathbf{L}_2}\Big] \tag{3.17}$$

from which it follows that

$$FF'' - \left(\frac{\beta+1}{2\beta+1}\right) F'^2 \geqslant -\Gamma FF' + 2F\Big[2\,|\,\mathscr{E}(0)| - 2\left\langle \mathbf{u}_0, \int_{-\infty}^{0} \mathbf{K}(-\tau)\,\mathbf{u}(\tau)\,d\tau \right\rangle_{\mathbf{L}_2}$$
$$- 2\int_0^t \left\langle \mathbf{u}(\tau), \int_{-\infty}^{\tau} \mathbf{K}_\tau(\tau-\lambda)\,\mathbf{u}(\lambda)\,d\lambda \right\rangle_{\mathbf{L}_2} d\tau$$
$$+ \left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\,\mathbf{u}(\tau)\,d\tau \right\rangle_{\mathbf{L}_2}\Big] \tag{3.18}$$

by virtue of hypothesis A1 relative to $\mathbf{K}(0)$. We now have the following estimates for the integrals appearing on the right-hand side of (3.18):

(i)
$$\left| \left\langle \mathbf{u}_0 , \int_{-\infty}^{0} \mathbf{K}(-\tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}_2} \right|$$
$$\leqslant \| \mathbf{u}_0 \|_{\mathbf{L}_2} \int_{-\infty}^{0} \| \mathbf{K}(-\tau)\|_{\mathscr{L}_s(\mathbf{H}_0^1;\mathbf{H}^{-1})} \| \mathbf{u}(\tau)\|_{\mathbf{H}_0^1}\, d\tau$$
$$\leqslant \gamma \| \mathbf{u}_0 \|_{\mathbf{H}_0^1} \sup_{[-t_h,0)} \| \mathbf{u} \|_{\mathbf{H}_0^1} \int_{-\infty}^{0} \| \mathbf{K}(-\tau)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\tau$$
$$\leqslant \gamma(\sup_{[-t_h, \infty)} \| \mathbf{u}(t)\|_{\mathbf{H}_0^1})^2 \int_{0}^{\infty} \| \mathbf{K}(\tau)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\tau$$
$$\leqslant \gamma N^2 \int_{0}^{\infty} \| \mathbf{K}(\tau)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\tau$$

therefore,

$$-2 \left\langle \mathbf{u}_0 , \int_{-\infty}^{0} \mathbf{K}(-\tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}} \geqslant -2\gamma N^2 \int_{0}^{\infty} \| \mathbf{K}(t)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\tau \tag{3.19}$$

(ii)
$$\left| \left\langle \mathbf{u}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}_2} \right|$$
$$\leqslant \| \mathbf{u}(t)\|_{\mathbf{L}_2} \int_{-\infty}^{t} \| \mathbf{K}(t-\tau)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})} \| \mathbf{u}(\tau)\|_{\mathbf{H}_0^1}\, d\tau$$
$$\leqslant \gamma(\sup_{[-t_h, \infty)} \| \mathbf{u}(t)\|_{\mathbf{H}_0^1})^2 \int_{-\infty}^{t} \| \mathbf{K}(t-\tau)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\tau$$
$$= \gamma(\sup_{[-t_h, \infty)} \| \mathbf{u}(t)\|_{\mathbf{H}_0^1})^2 \int_{0}^{\infty} \| \mathbf{K}(\rho)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\rho$$
$$\leqslant \gamma N^2 \int_{0}^{\infty} \| \mathbf{K}(t)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, dt$$

and, therefore, for $0 \leqslant t < \infty$,

$$\left\langle \mathbf{u}, \int_{-\infty} \mathbf{K}(t-\tau)\, \mathbf{u}(\tau)\, d\tau \right\rangle_{\mathbf{L}_2} \geqslant -\gamma N^2 \int_{0}^{\infty} \| \mathbf{K}(t)\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})}\, d\tau. \tag{3.20}$$

Finally, we have

(iii)
$$\left| \int_{0}^{t} \left\langle \mathbf{u}(\tau), \int_{-\infty}^{\tau} \mathbf{K}_\tau(\tau-\lambda)\, \mathbf{u}(\lambda)\, d\lambda \right\rangle_{\mathbf{L}_2} d\tau \right|$$
$$\leqslant \int_{0}^{t} \left| \left\langle \mathbf{u}(\tau), \int_{-t_h}^{\tau} \mathbf{K}_\tau(\tau-\lambda)\, \mathbf{u}(\lambda)\, d\lambda \right\rangle_{\mathbf{L}_2} \right| d\tau$$

$$\leqslant \int_0^\infty \left(\| \mathbf{u}(\tau)\|_{\mathbf{L}_2} \int_{-t_h}^{\tau} \| \mathbf{K}_\tau(\tau - \lambda)\|_{\mathscr{L}_s(\mathbf{H}_0{}^1, \mathbf{H}^{-1})} \| \mathbf{u}(\lambda)\|_{\mathbf{H}_0{}^1} \, d\lambda \right) d\tau$$

$$\leqslant \gamma(\sup_{[-t_h, \infty)} \| \mathbf{u}(t)\|_{\mathbf{H}_0{}^1})^2 \int_0^\infty \int_{-t_h}^{\tau} \| \mathbf{K}_\tau(\tau - \lambda)\|_{\mathscr{L}_s(\mathbf{H}_0{}^1, \mathbf{H}^{-1})} \, d\lambda \, d\tau$$

$$= \gamma(\sup_{[-t_h, \infty)} \| \mathbf{u}(t)\|_{\mathbf{H}_0{}^1})^2 \int_0^\infty \int_0^{\tau + t_h} \| \mathbf{K}_\rho(\rho)\|_{\mathscr{L}_s(\mathbf{H}_0{}^1, \mathbf{H}^{-1})} \, d\rho \, d\tau$$

$$\leqslant \gamma N^2 \int_0^\infty (\hat{\mathscr{K}}(\tau)|_0^{\tau + t_h}) \, d\tau$$

$$= \gamma N^2 \int_0^\infty \hat{\mathscr{K}}(\tau + t_h) \, d\tau$$

$$= \gamma N^2 \int_{t_h}^\infty \hat{\mathscr{K}}(\lambda) \, d\lambda \leqslant \gamma N^2 \| \hat{\mathscr{K}} \|_{L_1[0,\infty)} \cdot$$

where $\hat{\mathscr{K}}(\lambda) = \int \| \mathbf{K}_\lambda(\lambda)\|_{\mathscr{L}_s(\mathbf{H}_0{}^1 \mathbf{H}^{-1})} \, d\lambda$. Therefore, for $0 \leqslant t < \infty$,

$$-2 \int_0^t \left\langle \mathbf{u}(\tau), \int_{-\infty}^{\tau} \mathbf{K}_\tau(\tau - \lambda) \, \mathbf{u}(\lambda) \, d\lambda \right\rangle_{\mathbf{L}_2} d\tau \geqslant -2\gamma \cdot N^2 \| \hat{\mathscr{K}} \|_{L_1[0,\infty)} \cdot \tag{3.21}$$

Combining (3.18) with (3.19)–(3.21) then yields the estimate

$$\begin{aligned} &FF'' - \left(\frac{\beta + 1}{2\beta + 1}\right) F'^2 \\ &\quad \geqslant -\Gamma FF' + 2F[2 \mid \mathscr{E}(0)| - 3\gamma N^2 \{\| \mathscr{K} \|_{L_1[0,\infty)} + \| \hat{\mathscr{K}} \|_{L_1[0,\infty)}\}] \end{aligned} \tag{3.22}$$

which, in view of our hypothesis relative to $| \mathscr{E}(0)|$, implies the stated inequality, i.e., (3.2).

COROLLARY 1. *Under the same conditions which prevail in the theorem above*

$$\lim_{t \to \infty} \| \mathbf{u}(t)\|_{\mathbf{L}_2}^2 > \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2 \exp \left(\frac{2\langle \mathbf{u}_0 , \mathbf{v}_0 \rangle_{\mathbf{L}_2}}{\Gamma \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2} \right). \tag{3.23}$$

Proof. In (3.2), which is valid for all $\beta > 0$, we take the limit $\beta \to 0^+$ and obtain

$$FF'' - F'^2 \geqslant -\Gamma FF', \qquad 0 \leqslant t < \infty. \tag{3.24}$$

Direct integration of this differential inequality then yields the lower bound

$$F(t) \geqslant F(0) \exp \left[\left(\frac{F'(0)}{\Gamma F(0)} \right) (1 - e^{-\Gamma t}) \right], \qquad 0 \leqslant t < \infty \tag{3.25}$$

which in turn, implies that

$$\lim_{t\to+\infty} F(t) \geqslant F(0)\exp\left(\frac{F'(0)}{\Gamma F(0)}\right). \tag{3.26}$$

This last result is equivalent, via the definition of $F(t)$, to (3.23). Q.E.D.

A better lower bound and asymptotic estimate (as $t \to +\infty$) may be obtained with a little further effort; namely, we have

COROLLARY 2. *Under the same conditions which prevailed in the above theorem, it follows that for all $t > 0$, and any α, $\frac{1}{2} < \alpha < 1$,*

$$\|\mathbf{u}\|^2_{\mathbf{L}_2} \geqslant \|\mathbf{u}_0\|^2_{\mathbf{L}_2}\left[1 + \left(\frac{2(1-\alpha)\langle \mathbf{u}_0, \mathbf{v}_0\rangle_{\mathbf{L}_2}}{\Gamma\|\mathbf{u}_0\|^2_{\mathbf{L}_2}}\right)(1 - e^{-\Gamma t})\right]^{1/(1-\alpha)} \tag{3.27a}$$

so that, as $t \to +\infty$

$$\lim_{t\to+\infty} \|\mathbf{u}(t)\|^2_{\mathbf{L}_2} \geqslant \|\mathbf{u}_0\|^2_{\mathbf{L}_2}\left[1 + \frac{2(1-\alpha)\langle \mathbf{u}_0, \mathbf{v}_0\rangle_{\mathbf{L}_2}}{\Gamma\|\mathbf{u}_0\|^2_{\mathbf{L}_2}}\right]^{1/(1-\alpha)} \tag{3.27b}$$

Proof. For any $\alpha > 0$

$$[F^{(1-\alpha)}]''(t) = (1-\alpha)F^{-\alpha-1}(t)\,[F(t)F''(t) - \alpha F'^2(t)] \tag{3.28}$$

and from (3.2) with $\alpha = (\beta + 1)/(2\beta + 1)$

$$\begin{aligned}(1-\alpha)F^{-\alpha-1}[FF'' - \alpha F'^2] &\geqslant (1-\alpha)F^{-\alpha-1}[-\Gamma FF'] \\ &= -\Gamma(1-\alpha)F^{-\alpha}F'.\end{aligned} \tag{3.29}$$

Therefore for $\alpha = (\beta + 1)/(2\beta + 1)$

$$\begin{aligned}[F^{(1-\alpha)}]''(t) &\geqslant -\Gamma(1-\alpha)F^{-\alpha}F' \\ &= -\Gamma[F^{(1-\alpha)}]'(t).\end{aligned} \tag{3.30}$$

Let $G(t) = F^{(1-\alpha)}(t)$ and $H(t) = G'(t)$; then (3.30) implies that $H'(t) \geqslant -\Gamma H(t)$ and an integration produces

$$H(t) \geqslant H(0)\,e^{-\Gamma t} \leftrightarrow G'(t) \geqslant G'(0)\,e^{-\Gamma t}.$$

A second integration then yields

$$G(t) \geqslant G(0) + \frac{G'(0)}{\Gamma}[1 - e^{-\Gamma t}]$$

which is equivalent to

$$\begin{aligned} F^{(1-\alpha)}(t) &\geqslant F^{(1-\alpha)}(0) + \frac{(1-\alpha)\, F^{-\alpha}(0)\, F'(0)}{\Gamma}(1-e^{-\Gamma t}) \\ &= F^{(1-\alpha)}(0)\left[1 + \frac{(1-\alpha)\, F'(0)}{\Gamma F(0)}(1-e^{-\Gamma t})\right] \end{aligned} \tag{3.31}$$

from which the stated estimate (3.27a) follows after taking the $(1-\alpha)$th root on both sides of (3.31) and using the definition of $F(t)$; we note that (3.27b) follows directly from this last estimate and that $\alpha = (\beta+1)/(2\beta+1)$ takes on all values in the in the interval $(\frac{1}{2}, 1)$ for $\beta > 0$. Q.E.D.

Remark. Clearly as $\beta \to 0^+$, $\alpha \to 1$; taking the limit in (3.27b) as $\alpha \to 1$ and using the elementary fact that

$$\lim_{\lambda\to 0}[1+\lambda x]^{1/\lambda} = e^x$$

we recover (3.23) from (3.27b).

Remark. Clearly both (3.23) and (3.27b) imply that

$$\lim_{\Gamma\to+\infty}\ \lim_{t\to+\infty} \|\mathbf{u}(t)\|^2_{\mathbf{L}_2} \geqslant \|\mathbf{u}_0\|^2_{\mathbf{L}_2}$$

so that the $\mathbf{L}_2$ norm of $\mathbf{u}$ is bounded from below as $t\to+\infty$ even as the damping becomes arbitrarily large; this is the analogue, for the ill-posed integrodifferential initial-history value problem (2.1), of the asymptotic lower bound obtained in [15].

Remark. We comment here on some of the conditions imposed by the hypothesis of the theorem on the electromagnetic memory functions Φ and Ψ which appear in (1.5) and serve, therefore, to define the operators $\mathbf{N}$ and $\mathbf{K}(t)$; we have[4]

$$\begin{aligned} \langle \mathbf{v}, \mathbf{K}(0)\,\mathbf{v}\rangle_{\mathbf{L}_2} &= \int_{\tilde\Omega} v_i[\mathbf{K}(0)\,\mathbf{v}]_i\, d\mathbf{x} \\ &= \Psi(0)\int_{\tilde\Omega} v_i v_i\, d\mathbf{x} - \left(\frac{b_0}{a_0}\right)\Phi(0)\int_{\tilde\Omega} v_i\nabla^2 v_i\, d\mathbf{x} \\ &= \Psi(0)\,\|\mathbf{v}\|^2_{\mathbf{L}_2} - \left(\frac{b_0}{a_0}\right)\Phi(0)\left[\int_{\partial\tilde\Omega} v_i\frac{\partial v}{\partial x_j} n_j\, d\mathbf{x} - \int_{\tilde\Omega}\frac{\partial v_i}{\partial x_i}\frac{\partial v_i}{\partial x_j}\, d\mathbf{x}\right] \\ &= \Psi(0)\,\|\mathbf{v}\|^2_{\mathbf{L}_2} + \left(\frac{b_0}{a_0}\right)\Phi(0)\,\|\mathbf{v}\|^2_{\mathbf{H}_0^1} \end{aligned} \tag{3.32}$$

[4] To be consistent with the formulation of the initial-history boundary value problem in Section 1 we have, in fact, $\mathbf{v} = \mathbf{0}$ in $\tilde\Omega/\Omega$ in the computation below.

for any $\mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$. Therefore (hypothesis A1) $-\langle \mathbf{v}, \mathbf{K}(0)\,\mathbf{v}\rangle_{\mathbf{L}_2} \geqslant 0$, $\forall \mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$, iff

$$\ddot{\Psi}(0)\,\|\,\mathbf{v}\,\|_{\mathbf{L}_2}^2 + \left(\frac{b_0}{a_0}\right)\Phi(0)\,\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}^2 \leqslant 0. \tag{3.33}$$

If $\ddot{\Psi}(0) \geqslant 0$ then via the embedding of $\mathbf{H}_0^1(\tilde{\Omega})$, into $\mathbf{L}_2(\tilde{\Omega})$

$$\ddot{\Psi}(0)\,\|\,\mathbf{v}\,\|_{\mathbf{L}_2}^2 \leqslant \gamma^2\ddot{\Psi}(0)\,\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}^2$$

and (3.33) will be satisfied, for all $\mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega})$, provided

$$\gamma^2\ddot{\Psi}(0) + \left(\frac{b_0}{a_0}\right)\Phi(0) \leqslant 0 \leftrightarrow \Phi(0) \leqslant -\left(\frac{a_0}{b_0}\right)\gamma^2\ddot{\Psi}(0).$$

Thus, as far as hypothesis A1 goes, we have

$$\left\{\ddot{\Psi}(0) \geqslant 0,\, \Phi(0) \leqslant -\left(\frac{a_0}{b_0}\right)\gamma^2\ddot{\Psi}(0)\right\} \Rightarrow -\langle \mathbf{v}, \mathbf{K}(0)\,\mathbf{v}\rangle_{\mathbf{L}_2} \geqslant 0, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\tilde{\Omega}). \tag{3.34}$$

In view of (3.33), the same conclusion obtains if $\ddot{\Psi}(0) \leqslant 0$, $\Phi(0) \leqslant 0$. Also

$$\begin{aligned}
\|\,\mathbf{K}(t)\|_{\mathscr{L}_s(\mathbf{H}_0^2,\mathbf{H}^{-1})} &= \sup_{\mathbf{v}\in\mathbf{H}_0^1} \frac{|\langle \mathbf{v}, \mathbf{K}(t)\,\mathbf{v}\rangle_{\mathbf{L}_2}\,|}{\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}} \\
&= \sup_{\mathbf{v}\in\mathbf{H}_0^1} \frac{|\int_{\tilde{\Omega}} v_i[\mathbf{K}(t)\,v_i]\,d\mathbf{x}\,|}{\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}} \\
&= \sup_{\mathbf{v}\in\mathbf{H}_0^1} \frac{\left|\ddot{\Psi}(t)\,\|\,\mathbf{v}\,\|_{\mathbf{L}_2}^2 + \left(\frac{b_0}{a_0}\right)\Phi(t)\,\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}^2\right|}{\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}^2} \\
&\leqslant \sup_{\mathbf{v}\in\mathbf{H}_0^1}\left(\frac{|\,\ddot{\Psi}(t)|\,\|\,\mathbf{v}\,\|_{\mathbf{L}_2}^1}{\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1}^2}\right) + \left(\frac{b_0}{a_0}\right)\Phi(t) \\
&\leqslant \gamma^2\,|\,\ddot{\Psi}(t)| + \left(\frac{b_0}{a_0}\right)|\,\Phi(t)|\,.
\end{aligned} \tag{3.35}$$

Clearly, hypothesis A2 will then be satisfied if $\int_0^\infty |\,\ddot{\Psi}\,|\,dt < \infty$, and $\int_0^\infty |\,\Phi(t)|\,dt < \infty$, i.e.,

$$\{|\,\ddot{\Psi}\,| \in L_1[0, \infty),\, |\,\Phi\,| \in L_1[0, \infty)\} \to \mathscr{K} \in L_1[0, \infty). \tag{3.36}$$

A computation entirely analogous to (3.35) yields

$$\|\,\mathbf{K}_t\,\|_{\mathscr{L}_s(\mathbf{H}_0^1,\mathbf{H}^{-1})} \leqslant \gamma^2\,|\,\Psi^{(3)}(t)| + \left(\frac{b_0}{a_0}\right)|\,\dot{\Phi}(t)| \tag{3.37}$$

and, therefore, for hypothesis A3 we have

$$\begin{aligned} &\int \mid \Psi^{(3)} \mid dt \in L_1[0, \infty), \qquad \int \mid \dot{\Phi} \mid dt \in L_1[0, \infty) \\ &\int \mid \Psi^{(3)} \mid dt \mid_{t=0} = 0, \qquad \int \mid \dot{\Phi} \mid (t) \mid dt \mid_{t=0} = 0 \end{aligned} \tag{3.38}$$

$\Rightarrow \hat{\mathscr{K}} \in \mathbf{L}_1[0, \infty)$ with $\hat{\mathscr{K}}(0) = 0$.

Finally, for any $\mathbf{v} \in \mathbf{H}_0{}^1(\tilde{\Omega})$, we have

$$\begin{aligned} \langle \mathbf{v}, \mathbf{N}\mathbf{v} \rangle_{\mathbf{L}_2} &= \int_{\tilde{\Omega}} v_i [\mathbf{N}\mathbf{v}]_i \, d\mathbf{x} \\ &= \dot{\Psi}(0) \left[\int_{\tilde{\Omega}} c_0 v_i \nabla^2 v_i \, d\mathbf{x} - \int_{\tilde{\Omega}} v_i v_i \, d\mathbf{x} \right] \\ &= c_0 \dot{\Psi}(0) \left[\int_{\partial\tilde{\Omega}} v_i \frac{\partial v_i}{\partial x_j} n_j \, d\mathbf{x} - \int_{\tilde{\Omega}} \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x} \right] - \dot{\Psi}(0) \| \mathbf{v} \|_{\mathbf{L}_2}^2 \\ &= -\dot{\Psi}(0) \, [c_0 \| \mathbf{v} \|_{\mathbf{H}_0{}^1}^2 + \| \mathbf{v} \|_{\mathbf{L}_2}^2] \end{aligned} \tag{3.39}$$

therefore,

$$\begin{aligned} 2\mathscr{E}(0) &= \| \mathbf{v}_0 \|_{\mathbf{L}_2}^2 - \langle \mathbf{u}_0 , \mathbf{N}\mathbf{u}_0 \rangle_{\mathbf{L}_2} \\ &= \| \mathbf{v}_0 \|_{\mathbf{L}_2}^2 + \dot{\Psi}(0) \, [c_0 \| \mathbf{u}_0 \|_{\mathbf{H}_0{}^1}^2 + \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2] \\ &= \| \mathbf{v}_0 \|_{\mathbf{L}_2}^2 + \left(\frac{b_0}{a_0} \right) \| \mathbf{u}_0 \|_{\mathbf{H}_0{}^1}^2 + \dot{\Psi}(0) \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2 < 0 \end{aligned} \tag{3.40}$$

iff

$$\dot{\Psi}(0) < - \left[\| \mathbf{v}_0 \|_{\mathbf{L}_2}^2 + \left(\frac{b_0}{a_0} \right) \| \mathbf{u}_0 \|_{\mathbf{H}_0{}^1}^2 \right] \Big/ \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2 . \tag{3.41}$$

If $\dot{\Psi}(0)$ satisfies (3.41) then

$$\mid \mathscr{E}(0) \mid = \frac{1}{2} \left[\mid \dot{\Psi}(0) \mid \| \mathbf{u}_0 \|_{\mathbf{L}_2}^2 - \left(\| \mathbf{v}_0 \|_{\mathbf{L}_2}^2 + \left(\frac{b_0}{a_0} \right) \| \mathbf{u}_0 \|_{\mathbf{H}_0{}^1}^2 \right) \right] \tag{3.41}$$

and (3.1) is equivalent to requiring that

$$\mid \dot{\Psi}(0) \mid \geqslant \frac{1}{\| \mathbf{u} \|_{\mathbf{L}_2}^2} \left[3\gamma N^2 (\| \mathscr{K} \|_{L_1[0,\infty)} + \| \hat{\mathscr{K}} \|_{L_1[0,\infty)}) + \| \mathbf{v}_0 \|_{\mathbf{L}_2}^2 + \left(\frac{b_0}{a_0} \right) \| \mathbf{u}_0 \|_{\mathbf{H}_0{}^1}^2 \right] \tag{3.42}$$

where

$$
\begin{aligned}
\| \mathscr{K} \|_{L_1[0,\infty)} &\leq \gamma^2 \int_0^\infty |\ddot{\Psi}(t)|\, dt + \left(\frac{b_0}{a_0}\right) \int_0^\infty |\Phi(t)|\, dt \\
\| \hat{\mathscr{K}} \|_{L_1[0,\infty)} &\leq \gamma^2 \int_0^\infty \int_0^\tau |\Psi^{(3)}(\lambda)|\, d\lambda\, d\tau + \left(\frac{b_0}{a_0}\right) \int_0^\infty \int_0^\tau |\dot{\Phi}|\, d\lambda\, d\tau
\end{aligned}
\tag{3.43}
$$

by (3.35), (3.37), and the definitions of $\mathscr{K}(\cdot)$ and $\hat{\mathscr{K}}(\cdot)$.

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